

UNIVERSITY OF PENNSYLVANIA

Concerning the Cut Points of a Continuous
Curve when the Arc Curve, AB , Contains
Exactly N Independent Arcs

BY
NORMAN EBY RUTT

A Thesis

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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Concerning the Cut Points of a Continuous Curve when the Arc Curve, AB , Contains Exactly N Independent Arcs.*

BY NORMAN E. RUTT.

In 1925 Menger † introduced the term “curve” for those compact connected spaces which have the property that for every point of the set and every $\epsilon > 0$ there exists a neighborhood of diameter less than ϵ which contains the point and has a totally disconnected ‡ boundary. A point of such a set is said to be *regular* if it is possible to choose these neighborhoods in such a manner that its boundary is finite, while it is *of order n* if for every ϵ we can find the neighborhood of diameter less than ϵ whose boundary consists of exactly n points. In a subsequent paper he shows that in a regular curve, *i. e.*, a curve containing only regular points, there is corresponding to every point of order n a set of n arcs having the point in common and otherwise distinct.§ It should be noted that every regular curve of Menger is also a continuous curve,¶ that is, it is connected im kleinen at every point. Of course there are continuous curves which are neither regular nor even curves in the sense defined above.

In this paper is considered a plane continuous curve in which there are two points a and b which have the property that *there are in the continuous curve exactly n arcs from a to b which have no points other than a and b in common while it is impossible to find a similar set of $n + 1$ arcs from a to b .* Under these hypotheses it is proved that there are exactly n points in the curve whose omission separates a from b in the curve. This theorem provides a sort of converse to the theorem of Menger mentioned. The precise converse of the theorem of this paper is an easy consequence of the theorem itself, and

* Presented to the American Mathematical Society, May 7, 1927.

† Cf. K. Menger, “Grundzüge einer Theorie der Kurven,” *Mathematische Annalen*, Vol. 95 (1927), p. 277.

‡ A set of points is said to be *totally disconnected* if it contains no connected subsets other than single points.

§ See “Zur allgemeine Kurventheorie,” *Fundamenta Mathematicae*, Vol. 10, (1927), p. 98.

¶ For fundamental definitions and properties of continuous curves see R. L. Moore, “Report on Continuous Curves from the Viewpoint of Analysis Situs,” *Bulletin of the American Mathematical Society*, Vol. 29 (1923), pp. 289-302.

thus furnishes a proof of Menger's theorem for more general *plane continuous curves*. In the course of the argument very important use is made of the arc-curve $E(ab)$ of the continuous curve E with respect to the distinct points a and b , i. e., the set of all points $[X]$ of E such that for every point x of the set there is an arc of E from a to b which contains x .*

The statement of the principal theorem of this paper is due to Professor J. R. Kline, and it has been with his advice and direction that the proof has been constructed.

1. GENERAL THEOREM. *If there exists an integer n such that for two given points a and b of a continuous curve E there are arcs ax_ib ($i = 1, 2, \dots, n$) of E no two of which have any points except a and b in common, and such that by no means is it possible to construct $n + 1$ arcs like these from a to b , then there exists a set of n distinct points $p_1, p_2, \dots, p_n$ of E such that $E - (p_1 + p_2 + \dots + p_n)$ is the sum of two mutually separated sets E_a and E_b , E_a containing a and E_b containing b .*

Notation and definitions. Let F be the arc curve of the point set $a + b$ with respect to E , that is, let F be the set of all arcs that can be drawn in E from a to b . It is a continuous curve every complementary domain of which has a simple closed curve as its boundary.† Let a particular set of n arcs ax_ib satisfying the hypotheses of the theorem be $M_1, M_2, \dots, M_n$ (such a set will hereafter be referred to as *independent*) and let these be ordered in such a way that the simple closed curve formed by M_{i-1} and M_{i+1} contains in its interior (M_i) , that is $M_i - (a + b)$, but not (M_j) , where $j = 1, 2, \dots, n$ and $i = 2, 3, \dots, (n - 1)$ and $j \neq i$. Let M_{ij} be the simple closed curve $M_i + M_j$, and let its interior be I_{ij} . Let F_{ij} represent $M_{ij} + F \times I_{ij}$. F_{ij} is a continuous curve;‡ the arc-curve of $(a + b)$ with respect to F_{ij} is F_{ij} .§ Let any complementary domain of F_{jj+1} whose boundary has at least one point on each of the arcs (M_j) and (M_{j+1}) be called a *cross-domain* or *c-domain* of F .

* This notion has been introduced by W. L. Ayres in a paper which has been accepted for publication by the *Transactions of the American Mathematical Society*. Ayres proves that the arc curve $E(ab)$ is itself a continuous curve. For an abstract of this paper see the *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 410. As this paper is as yet unpublished the numerous references to it throughout these pages are somewhat incomplete.

† W. L. Ayres, "Concerning the Arc-curves and Basic Sets of a Continuous Curve"; see remarks above in introduction.

‡ H. M. Gehman, "Concerning the Subsets of a Plane Continuous Curve," *Annals of Mathematics*, Vol. 27, 1925, No. 1, p. 34.

§ W. L. Ayres, "Concerning the Arc-curves and Basic Sets of a Continuous Curve."

In what follows ordinarily c -domains will be represented by small Greek letters γ, δ , their boundaries by the corresponding capitals Γ, Δ .

I(a) THEOREM. *At least one of the complementary domains of F_{i+1} is a cross-domain of F .*

Proof: Consider the set $F_{12} - M_{12}$. If this is vacuous the theorem is obvious as then I_{12} is a c -domain. Since F_{12} and M_{12} are both continuous curves and $M_{12} \subset F_{12}$, if $F_{12} - M_{12}$ is not vacuous, it consists of a certain number of maximal connected parts. Suppose that one of these, say K , has limit points both upon (M_1) and upon (M_2) . If both a and b are also limits of K there must in K be an arc from a to b .^{*} Either this is an $n + 1$ st arc from a to b contradicting hypothesis, or it is one of the arcs M_j , $j > 2$, contrary to the convention used in ordering $M_1, \dots, M_n$. Suppose then that a is not a limit point of K . As the limit points of K form a closed set with representative points both upon (M_1) and upon (M_2) there is a first k_1 on the one and a first k_2 on the other. Now $M_{12} - (k_1 + k_2)$ is disconnected into (k_1ak_2) and (k_1bk_2) . The arcs (k_1ak_2) and (k_1bk_2) will then occur in the same maximal connected subset of $F_{12} - (k_1 + k_2)$ only if some maximal connected subset of $F_{12} - M_{12}$ has limit points on both (k_1ak_2) and (k_1bk_2) . Suppose one of these, say L , has one, l_a , in (k_1ak_2) and one, l_b , in (k_1bk_2) . Draw an arc (k_1xk_2) in K and one (l_ayl_b) in L , as these must intersect interior to I_{12} and thus K be identically L , a contradiction arises which shows that $F_{12} - (k_1 + k_2)$ is disconnected. But then k_1 and k_2 are on the boundary of a certain bounded complementary domain of F_{12} .[†]

Suppose then that no maximal connected subset of $F_{12} - M_{12}$ has a limit point on (M_1) and another on (M_2) . As before $F_{12} - (a + b)$ is disconnected and so a and b lie on the boundary of some bounded complementary domain D of F_{12} . The boundary of this domain is a subset of F_{12} and is a simple closed curve which is divided into two arcs (ayb) and (azb) by the points a and b . As neither of these arcs has any points exterior to M_{12} it is clearly impossible for them both to have points on either (M_1) or (M_2) , or for one to have points on both (M_1) and (M_2) . But if (ayb) has no point on (M_1) it must have at least one on (M_2) as otherwise it is an $(n + 1)$ st arc from a to b in F . Then similarly (azb) has a point on (M_1) . Thus as one of the two arcs has a point on (M_1) and the other one on (M_2) the domain

^{*} R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7, p. 342.

[†] G. T. Whyburn, Abstract of an unpublished paper in the *Bulletin of the American Mathematical Society*, Vol. 33, p. 388, No. 10.

which they bound is a cross-domain of F . The same result of course follows when (ayb) has a point on (M_1) .

The theorem is thus true for F_{12} . But as the argument used in the proof involved only F_{12} itself it evidently applies immediately to each set F_{jj+1} ($j = 2, 3, \dots, n-1$).

Notation and definitions. Let δ be any c -domain of F_{jj+1} , that is of F in I_{jj+1} . Its boundary Δ has a first point a_j on M_j and also a last b_j , likewise it has a first and last a_{j+1} and b_{j+1} on M_{j+1} . It may happen, (1) that $a_j \equiv a \equiv a_{j+1}$, (2) that $b_j \equiv b \equiv b_{j+1}$, (3) that $a_j \equiv a \equiv a_{j+1}$ and $b_j \equiv b \equiv b_{j+1}$. In these cases it will be convenient still to speak of these coincident pairs as if they were distinct points.

The symbols $a_j x a_{j+1}$ and $b_j y b_{j+1}$ each represent two arcs or points of Δ ; of these one from the first pair and one from the second have at most end points in common, and these two will be called the a - and b -arcs of δ , Δ , even although they may reduce to points. If γ and δ are c -domains of F_{jj+1} and γ precedes δ these two will be said to be *adjacent* when the b -arc of Γ , the a -arc of Δ , and the subarcs of M_j and M_{j+1} which their ends intercept either fail to form a simple closed curve or form one whose interior contains no c -domain of F .

I(b) THEOREM. *If γ and δ are adjacent c -domains of F_{jj+1} , γ preceding δ , then from a point of the b -arc of Γ in I_{jj+1} to a point of the a -arc of Δ in I_{jj+1} there is an arc of $F_{jj+1} - M_{jj+1}$ having only an end point in common with either Γ or Δ .*

Proof: The argument will be given only for F_{12} . Let the a - and b -arcs in question be $b_1 y b_2$ and $a_1 z a_2$. If $(b_1 y b_2) \times (a_1 z a_2) \neq 0$ any point of the intersection will be considered to satisfy the theorem. If this product is zero the two arcs involved in it form with the arcs of (M_1) and (M_2) which their ends intercept a simple closed curve C with interior I . If I contains no constituent of F with limits on both $(b_1 y b_2)$ and $(a_1 z a_2)$ by an argument like that in I(a) I contains a c -domain of F . Hence the theorem. Clearly the theorem is similarly true in the special cases $\gamma + \Gamma = a$, and $\delta + \Delta = b$.

I(c) THEOREM. *If the c -domains of F_{jj+1} form an infinite set this set must have at least one limit point not already a limit point of some finite subset of the set, and only a a or b can be such a limit point.*

Proof: That one such limit must exist is evident from the fact that the domains in question form a bounded infinite set. Suppose c is one distinct from a and b . At least one of the distances c to (M_j) , c to (M_{j+1}) is not zero; suppose the first is ϵ . Let c be the center of a circle of radius $\epsilon/2$. Within

this circle an infinite number of the c -domains in question must have points, and as each of these must also have a boundary point on (M_j) each c -domain of this infinite number must have diameter greater than $\epsilon/2$, an evident contradiction.

It may also be noted that if a is one of the limits of the c -domains of F_{jj+1} it can not be a boundary point of any c -domain of F_{jj+1} , for if it is a point of Δ and if p_j and p_{j+1} are points of Δ on (M_j) and (M_{j+1}) , $(p_j x p_{j+1})$ being an arc of δ , $p_j y b$ one on M_j and $p_{j+1} z b$ one of M_{j+1} , then a is exterior to the simple closed curve $p_j x p_{j+1} z b y p_j$ while all but δ of the set of c -domains of F_{jj+1} of which a is a set limit are interior to this curve. It is clear that the theorem may be stated equally truly for the boundaries of the c -domains of F_{jj+1} .

Notations and definitions. Owing to I(c) the c -domains of F_{jj+1} may be represented as $\dots, \delta_{-m}, \dots, \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \dots, \delta_m, \dots, \delta_0$ having been

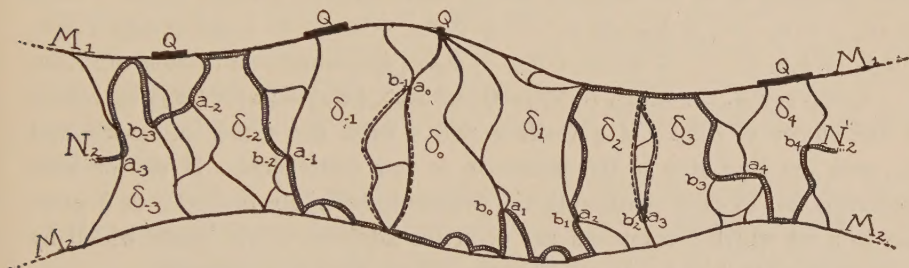


FIG. 1.

The heavy line is N_2 . From the sum have been omitted the dotted loupes.

selected at random. The i c -domains $\gamma_{jj+1}, \dots, \gamma_{j+i-1j+i}$ will be said to form a chain of F_{jj+1} from M_j to M_{j+i} if (1) one and only one of them is a complementary domain of each of the curves $F_{jj+1}, \dots, F_{j+i-1j+i}$; and (2) Γ_{q-1q} has at least one point of (M_q) in common with Γ_{qq+1} ($q=j+1, \dots, j+i-1$). $\Gamma_{jj+1}, \dots, \Gamma_{j+i-1j+i}$ will be referred to as links of the chain, and the points of Γ_{jj+1} on (M_j) and of $\Gamma_{j+i-1j+i}$ on (M_{j+i}) as end points of the chain.

I(d) THEOREM. If Q is any subset of (M_j) there are in F_{jj+1} , i independent arcs $N_{j+1}, N_{j+2}, \dots, N_{j+i}$ from a to b , F_{t-1t} containing N_t ($t=j+1, \dots, j+i$); where $N_{j+1} \times Q = 0$, and $N_t \times (M_t)$ is a subset of the end points on M_t of the chains of F_{jt} which have at least one opposite end point among the points of Q .

For the sake of simplifying the subscripts this theorem without loss of generality will be proved for F_{11+i} . The proof will be by induction. During the first part refer to fig. 1.

Proof: Let the set of all the c -domains of F in F_{12} be represented by $\sum_i \delta_i = \dots + \delta_{-s} + \dots + \delta_0 + \dots + \delta_t + \dots$ and the set of their boundaries by $\sum_i \Delta_i = \dots + \Delta_{-s} + \dots + \Delta_0 + \dots + \Delta_t + \dots$. Let $\{\Delta\}$ be the subset of $\sum_i \Delta_i$ consisting of all and only those for which $\Delta_i \times Q \neq 0$. If a is not a limit of the set $\sum_i \delta_i$ it will be convenient to consider it the first of the set $\sum_i \Delta_i$, under similar circumstances to consider b the last; under this convention $\sum_i \Delta_i$ will necessarily contain three or more members. If Δ_{i-1} and Δ_i are adjacent elements of $\sum_i \Delta_i$ let $b_{j-1}a_j$ be the arc of $I(b)$ (of course possibly $b_{j-1}a_j$ represents a point, even maybe a or b) for every j , and let similarly a_jb_j be that arc of Δ_j for which $a_jb_j \times M_1 = 0$ if $\Delta_j \times Q \neq 0$ otherwise for which $a_jb_j \times M_1 \neq 0$. Now the sum $a_{j-1}b_{j-1} + b_{j-1}a_j + a_jb_j$ is either an arc from a_{j-1} to b_j or fails to be one only because $(a_{j-1}b_{j-1}) \times (a_jb_j) \neq 0$. In the latter case, however, a proper subset of $a_{j-1}b_{j-1} + a_jb_j$ is an arc from a_{j-1} to b_j and the sum above will be then supposed to represent this arc; the same meaning is to be assigned to the more extensive sums below. Thus $b_{-r-1}a_{-r-1} + a_{-r-1}b_{-r} + b_{-r}a_{-r+1} + \dots + b_{-2}a_{-1} + a_{-1}b_{-1} + b_{-1}a_0 + a_0b_0 + b_0a_1 + a_1b_1 + b_1a_2 + \dots + b_{s-1}a_s + a_sb_s + b_s a_{s+1}$ is a simple continuous arc from b_{-r-1} to a_{s+1} say N_2 for which $N_2 \times Q = 0$ and $N_2 \times (M_2) = \{\Delta\} \times (M_2)$ by reason of the choice of a_jb_j . If a finite r and s exist for which $\Delta_{-r-1} \equiv a$ and $\Delta_{s+1} \equiv b$ the first step in the induction is now established. If one or both subscripts fail to exist continuing the process indefinitely toward a or b gives rise to a set which becomes an arc upon the addition of its limits which by $I(c)$ are at most a or b . In all cases then is the first step complete.

Assuming now that in F_{1k} from a to b there are $k-1$ independent arcs $N_2, \dots, N_k$ where $1 < k < i$, $N_2 \times Q = 0$, and $N_k \times (M_k)$ is a subset of the ends on (M_k) of those chains of F_{1k} among the ends on (M_1) of each of which there is at least one point of Q , the same thing for one greater subscript is to be proved. Let $\sum_i \gamma_i$ represent the set of c -domains of F_{kk+1} , and let $Q_k = N_k \times (M_k)$. By the first part of the proof there is in K_{kk+1} from a to b an arc N_{k+1} for which $N_{k+1} \times Q_k = 0$, hence an arc independent of $N_2, \dots, N_k$, and for which $N_{k+1} \times (M_{k+1})$ is a subset of $\{\Gamma_{kk+1}\} \times (M_{k+1})$ where $\{\Gamma_{kk+1}\}$ represents the subset of all the boundaries Γ_i of domains of $\sum_i \gamma_i$ for which $\Gamma_i \times Q_k \neq 0$. But if $\Gamma_i \times Q_k \neq 0$ then $\Gamma_i \times (M_k)$ includes an end of some chain from (M_1) to (M_k) and γ_i is a last domain of a chain from (M_1) to (M_{k+1}) so $\Gamma_i \times (M_{k+1})$ consists of ends on (M_{k+1}) of chains of F_{1k+1} whose ends on (M_1) include at least one point of Q for each chain, since the points of Q_k are the ends of similar chains of F_{1k} . It is to be noted that this construction makes $N_i \times (M_i)$ not a proper subset of the end points on M_i of the chains specified. The induction, however, is complete.

I(e) COROLLARY. If $Q_{t+1} = (M_{t+1}) \times \{\Gamma_{tt+1}\}$ ($t = 1, 2, \dots, i-1$) then $Q_{t+1} = (M_{t+1}) \times N_{t+1}$ and if $\{\bar{\Gamma}_{tt+1}\} = \sum_i \Gamma_t - \{\Gamma_{tt+1}\}$ then

$$N_{t+1} \times (M_t) = \{\bar{\Gamma}_{tt+1}\} \times (M_t) \subset (M_t) - Q_t.$$

The construction used above demonstrates this.

I(f) COROLLARY. In F_{1n} there is at least one chain from (M_1) to (M_n) .

For suppose not and apply I(d) to F_{1n} with $Q = (M_1)$. Then in F_{1n} there are $n+1$ independent arcs from a to b , namely $M_1, N_2, N_3, \dots, N_{n-1}, N_n$, and M_n as clearly $(M_1) \times N_2 = 0$ and $N_n \times (M_n) = 0$.

Remarks: The general theorem is now proved, except for the final steps in § 5, in case both a and b are on the simple closed curve which forms the outer boundary of F ; for in this case M_1 and M_n may always be chosen as the two arcs into which a and b divide this curve and the statements I(a) to I(f) inclusive still hold. The unbounded complementary domain of F together with any chain of F_{1n} existing by I(f) constitute the set of n domains employed in § 5. In §§ 2, 3, 4 it will accordingly be assumed that either a or b is not on the outer boundary of F . By transformations of simple sorts some of the cases in which this is true can be reduced to the case already treated, but as these are special it seems not worth while to treat them separately.

The points of F_{1n} which are either end points of chains of F_{1n} or are points common to two links of some such chain are of special importance and will be called *invariant*. As already mentioned, if $F_{1n} \equiv F$ the argument of § 5 applies without any further discussion with the consequence that the set of invariant points of F_{1n} is a subset of every set of n independent arcs of F_{1n} from a to b .

2. An analysis of the non-vacuous set $F - F_{1n}$ must now be made. Let it be G ; it consists of a set of maximal connected sets of which at most a finite number are of diameter greater than any assigned positive number ϵ ,* and each of which has at least two limit points in M_{1n} .† Let $G = H + I + J + K$ where, H consists of those constituents of G all whose boundary points are in either (M_1) or (M_n) but not both, and none of which contains an arc which with an arc of (M_1) (or of (M_n)) forms a simple closed curve having I_{1n} in its interior (the sets making up H are at most countably infinite); J consists of sets at most finite in number differing in no respect from those in H except that a simple closed curve as described in connection

* W. L. Ayres, "Concerning Continuous Curves and Correspondences," *Annals of Mathematics*, Second Series, Vol. 28, No. 3, 1927, July, p. 396.

† W. L. Ayres, "Concerning Arc Curves and Basic Sets of a Continuous Curve."

with H now exists containing I_{1n} in its interior; I is made up of all those constituents of G among whose boundary points are a or b but obviously not both, it consists of subsets A , all whose maximal connected parts have a as limit, and B , similarly defined; K is composed of all those constituents of G not already included in I and having at least one limit on each of (M_1) and (M_n) .

II(a) THEOREM. *The set $F_{1n} + H$ possesses all the properties of F_{1n} that were derived in section 1.*

Let any maximal connected part of H be H_i . Now $\overline{H}_i$, the set plus its limit points, is a continuous curve.* As F_{1n} is a continuous curve and as for every i , $\overline{H}_i \times F_{1n} \neq 0$, $F_{1n} + \sum_i \overline{H}_i$ is a continuous curve.† But since $\sum_i \overline{H}_i - H \subset F_{1n}$, $F_{1n} + \sum_i \overline{H}_i = F_{1n} + H$ and so $F_{1n} + H$ is also a continuous curve. The arc curve of $a + b$ with respect to $F_{1n} + H$ is $F_{1n} + H$, so the outer boundary of $F_{1n} + H$ is a simple closed curve. This simple closed curve must have a point on each of (M_1) and (M_n) as otherwise H must have contained constituents properly belonging to J or K . Let now $H + M_{1n}$ be inverted with center of inversion a point of I_{1n} , and M_{1n} as invariant simple closed curve. The unbounded complementary domain of $F_{1n} + H$ becomes a bounded cross-domain of $\Pi(M_{1n} + H)$ and as H had no constituent with limit points on both (M_1) and (M_n) , $\Pi(M_{1n} + H) - M_{1n}$ has no constituent with limit points in both (M_1) and (M_n) . Thus as in I(a) a and b are on the boundary of a bounded complementary domain of $\Pi(M_{1n} + H)$ which is also a c -domain of it. When a c -domain of this type exists it is clearly the only possible c -domain and so must be the inverse of the unbounded complementary domain of $F_{1n} + H$ which accordingly has a boundary containing a and b . The two arcs into which a and b divide this boundary may be taken as M_1 and M_n . The theorem is thus proved. Hereafter F will represent $F_{1n} + H$ so that $G = I + J + K$.

If $A \neq 0$, it has last points i_{1a} and i_{na} on M_1 and M_n respectively perhaps not both different from a but clearly both different from b . With regard to B there are similar points.

II(b). *A contains arcs (axi_{1a}) , (axi_{na}) and B contains similar arcs.*

For a is a limit of every constituent of A and if i_{1a} were not a limit of some particular one there would be a contradiction.

* See second footnote referred to in section 1 and in the theorem there used read "exterior" as well as "interior."

† H. M. Gehman, "Some Conditions under which a Continuum is a Continuous Curve," *Annals of Mathematics*, Second Series, Vol. 27, p. 381.

II(c). All the limits of J occur on an arc of (M_1) [or (M_n)] which except perhaps at its ends contain no limits of I or K .

II(d). The only limit points, not already limits of a particular constituent, which K can have are a and b .

The set K needs to be discussed in detail. Obviously the limits of any constituent of it on (M_1) or (M_n) occupy an interval whose interior points are none of them limits of any other constituent. In every K_j draw an arc from a limit on (M_1) to one on (M_n) . This arc forms with an arc of M_{1n} going through a a simple closed curve whose interior will be called I_j . It may happen for some arcs in K_j that I_j contains I_{1n} and for some not; clearly only one constituent of K_j can have this property, if such a constituent exists it will be represented by K_0 . For each of the remaining constituents K_j of K either I_j never contains I_{1n} in which case j will be considered negative or I_j always contains I_{1n} in which case j will be positive. If i and j are negative and $I_i \subset I_j$ then let i and j be integers such that $i < j$, whereas if i and j are positive and $I_i \supset I_j$ let i and j be integers and $i < j$. In view of II(d) and obvious facts, this representation is possible and may be taken as $\dots, K_{-i}, \dots, K_{-2}, K_{-1}, K_0, K_1, K_2, \dots, K_i, \dots$, where of course K_0 may be vacuous. Precedence among the constituents of K will hereafter be determined by referring to their subscripts.

The constituent K_j has on (M_1) and (M_n) first limit points a_{1j} and a_{nj} and last limit points b_{1j} and b_{nj} . On (M_1) the interval $b_{1j}a_{1j+1}$ will be called C_j , on (M_n) the interval $b_{nj}a_{nj+1}$ will be called D_j , and C_j, D_j will be referred to as a pair of *corresponding intervals*. If $K_0 = 0$ C_0 and C_{-1} will represent the same interval, D_0 and D_{-1} likewise. If $A \neq 0$ there is a first constituent of K say K_{-l} and a first pair of corresponding intervals $C_{-l-1} = i_{1a}a_{1,-l}$, and $D_{-l-1} = i_{na}a_{n,-l}$, one but not both of which may begin with a . Even when A is vacuous there may of course be a first K_{-l} with both C_{-l-1} and D_{-l-1} beginning at a . Relations between K and B of course present no features not possible between K and A . Obviously C_j and D_j may reduce to points but will nevertheless be referred to as intervals. When a (or b) is end of an interval the interval will be assumed to lack this point.

II(e). Any pair of corresponding intervals is on the boundary of a common complementary domain of F .

Assuming K and J vacuous the theorem is obvious as then $i_{1a}i_{1b} = C_0$ and $i_{na}i_{nb} = D_0$ are the only corresponding intervals and are on the outer boundary of F . If $J = 0$ but $K \neq 0$ invert $F - F \times I_{1n}$ from any point of I_{1n} leaving M_{1n} invariant and getting a set $\bar{F}$. By the same argument as that

of the first case of I(a) any point c of C_j and any point d of D_j are on the boundary of a common bounded complementary domain of $\bar{F}$, and any other point c' of C_j can fail to be on the same boundary only if a constituent of $\bar{F} - M_{1n}$ has a limit point interior to C_j , which is ridiculous; similarly any point d' of D is on the boundary of this same domain. Hence C_j and D_j are on the boundary of a common complementary domain of F . If the arc of II(c) occurs on one of a pair of corresponding intervals as it may if $J \neq 0$ it can occur only on C_0 or D_0 and on one of these only if $K_0 = 0$. In this case it again separates $C_0 = C_{-1}$ (or $D_0 = D_{-1}$) into two distinct sets both of which are easily proved to be on the boundary of the same domain as $D_0 = D_{-1}$ (or $C_0 = C_{-1}$).

The relation of the chains of F_{1n} to the pairs of corresponding intervals is of critical importance. Let C and D be any corresponding pair. If at least one chain of F_{1n} has ends on both C and D , these two are said to be *joined*. If no chain has ends on C (or D) then C (or D) is said to be *free*. If the only chains having ends on C (or D) have opposite ends all preceding D (or C) then C (or D) is said to be *a-joined*; if all following, C (or D) is said to be *b-joined*; if both preceding and following, C (or D) is said to be *ab-joined*. The following is a complete list of the possible relations of C and D to the chains of F_{1n} :

- | | |
|------------------------------|---------------------------------|
| 1 C free, D free | 10 C a-joined, D a-joined |
| 2 C free, D a-joined | 11 C a-joined, D ab-joined |
| 3 C free, D b-joined | 12 C b-joined, D b-joined |
| 4 C free, D ab-joined | 13 C b-joined, D ab-joined |
| 5 C a-joined, D free | 14 C ab-joined, D a-joined |
| 6 C b-joined, D free | 15 C ab-joined, D b-joined |
| 7 C ab-joined, D free | 16 C ab-joined, D ab-joined |
| 8 C a-joined, D b-joined | 17 C and D joined. |
| 9 C b-joined, D a-joined | |

The object of the next two sections is to show that the fundamental assumptions of the theorem imply that at least one pair of corresponding intervals is joined (relation 17). Section 3 is devoted to some preliminary considerations.

3. In any chain of F_{1n} an arc may be drawn from an end of the chain on (M_1) to one on (M_n) by way of domains of the chain with the exception of precisely n points one on each of the arcs (M_j) ($j = 1, \dots, n$) necessarily invariant. Such an arc will be called an *arc of the chain*. Two important arcs may also be drawn on the boundaries of domains of the chain.

Let an arc be drawn from the last end of the chain on M_1 to the last end on M_n so as to contain the b -arc of every link of the chain, this will be called the b -boundary arc of the chain. It may reduce to point b . The a -boundary arc is similarly defined. Special cases of no significance arise if more than one domain of the chain has a or b as limit.

III(a) THEOREM. *If on M_1 there is the order $a c_1 d_1 b$ and on M_n a $d_n c_n b$ and c_1, c_n are ends of a chain $\gamma_1, \dots, \gamma_{n-1}$, d_1, d_n of a chain $\delta_1, \dots, \delta_{n-1}$; then c_1, d_n are ends of a chain, also c_n, d_1 .*

Proof: Construct an arc $c_1 c_n$ of $\gamma_1, \dots, \gamma_{n-1}$ and one $d_1 d_n$ of $\delta_1, \dots, \delta_{n-1}$. Owing to the given order of their end points these two arcs intersect in I_{1n} . Counting from c_1 let p be the first common point. If p is interior to some curve M_{jj+1} , ($j = 1, 2, \dots, (n-1)$) then γ_j and δ_j coincide as p is interior to both; accordingly $\gamma_1, \dots, \gamma_{j-1}, \gamma_j \equiv \delta_j, \delta_{j+1}, \dots, \delta_{n-1}$ and $\delta_1, \dots, \delta_{j-1}, \delta_j \equiv \gamma_j, \gamma_{j+1}, \dots, \gamma_{n-1}$ are the chains fulfilling the requirements. If p is on M_j then p is in each of the sets $\Delta_{j-1}, \Delta_j, \Gamma_{j-1}$, and Γ_j . But then $\gamma_1, \dots, \gamma_{j-1}, \delta_j, \dots, \delta_{n-1}$ and $\delta_1, \dots, \delta_{j-1}, \gamma_j, \dots, \gamma_{n-1}$ answer the theorem.

This theorem shows that cases 10 to 16, inclusive, at the end of section 2 all reduce to 17 and from now on will be omitted. A common feature of cases 1 to 7 inclusive is the occurrence of at least one free interval. Let us consider the situation.

Any free interval must be a subset of some maximum free segment at least one end of which (but not necessarily both, as one may be a or b) is an end of a chain. Let $(l_1 f_1)$ be a maximum free segment with $f_1 = b$. Let γ_1 be the last c -domain of F_{1n} which is a link of a chain having l_1 on Γ_1 . If Γ_1 contains b complete the chain $\gamma_1, \dots, \gamma_{n-1}$ in any convenient way. If not let γ_2 be the last c -domain of F_{23} which is part of a chain from $al_1 \subset M_1$ to (M_n) for which $\Gamma_1 \times \Gamma_2 \neq 0$. Either Γ_2 contains b in which case complete the chain $\gamma_1, \gamma_2, \dots, \gamma_{n-1}$ in any convenient way, or it does not. In this case let γ_3 be the last c -domain of F_{34} which is part of a chain from al_1 to (M_n) with $\Gamma_2 \times \Gamma_3 \neq 0$. If b is on Γ_3 complete the chain at random, if not repeat the process above, and so on until a link of a chain is thus found with b on its boundary, or a complete chain consisting of last possible links is found from l_1 to an extreme point l_n on (M_n) . Draw the b -boundary arc of the chain above until it reaches b (call it then $l_1 b$) or until it reaches (M_n) (call it then $l_1 l_n$).

If l_1 is a , then a similar construction working toward a results either in an arc $f_1 f a$ or an arc $f_1 f f_n$. If $a \neq l_1$ and $f_1 \neq b$ then either l_1 and f_1 are on the boundary of some common c -domain of F_{12} or there is from l_1 an arc which is b -boundary arc of a chain consisting of last possible domains to an

extreme point l_n on (M_n) , and from f_1 to f_n (an extreme point of (M_n) toward a) an arc which is a -boundary arc of a chain of first possible domains.

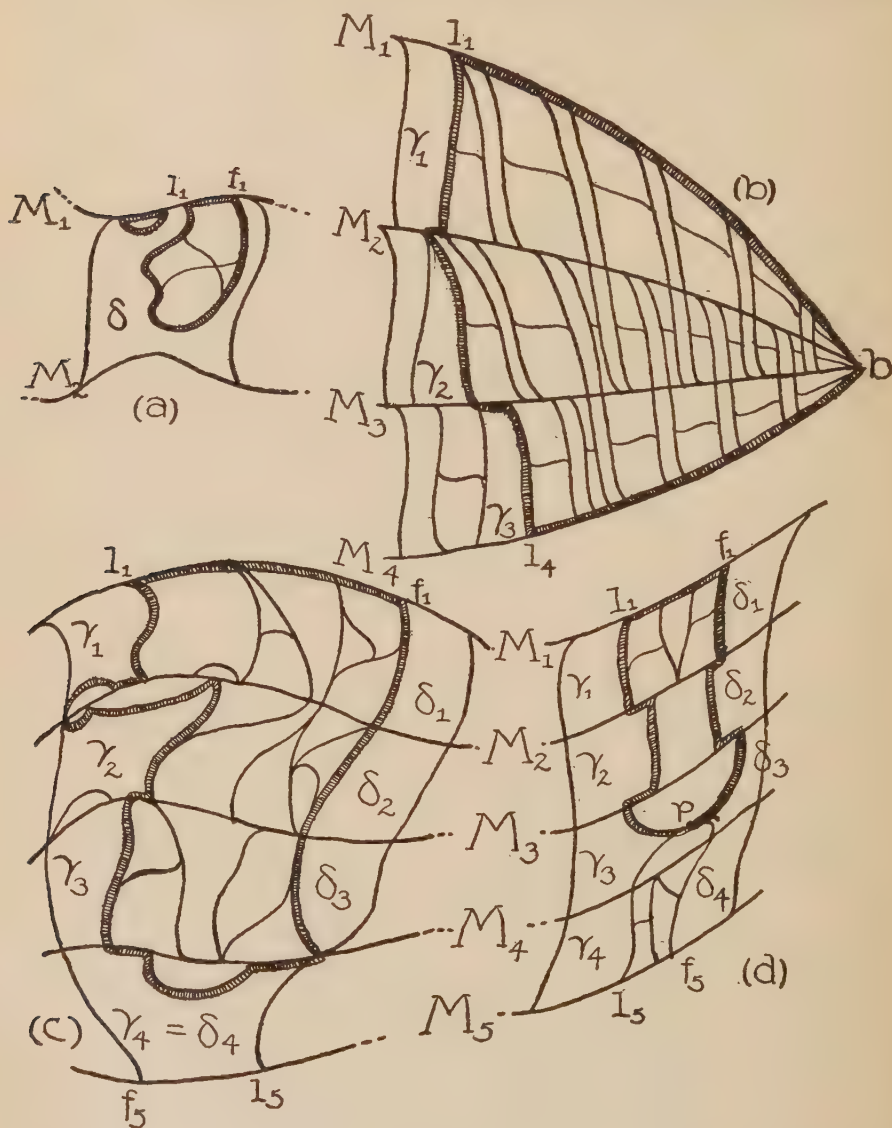


FIG. 2.

Pockets are enclosed by heavy line.

The arcs $l_1 l_n$ and $f_1 f_n$ may not intersect but if they do proceeding from l_1 call the first common point p and in the sum of the arcs draw the arc $l_1 p f_1$, where possibly $l_n = p = f_n$.

Now by means of the minor * arcs of M_{1n} complete the following simple closed curves; (1) if l_1 and f_1 are on the same Δ join them along the minor arc * of Δ and the minor arc of M_{1n} , fig. 2(a); (2) if there is only one chain and arc, $l_1 l b l_1$, $l_1 l_n b l_1$, $f_1 f a f_1$, $f_1 f f_n a f_1$, fig. 2(b); (3) if there are two arcs intersecting, $l_1 p f_1 l_1$, fig. 2(c); (4) if there are two arcs not intersecting, $l_1 l_n f_n l_1$. These curves and the subsets of F_{1n} which they enclose will be called *pocket subsets* or *pockets* of F_{1n} determined by $(l_1 f_1)$. Clearly maximum free segments on M_n as say $(l_n f_n)$ will determine similar sets in connection with it. Evidently the boundary of any pocket of F_{1n} with respect to F_{1n} consists of a finite number of invariant points unless the point p happens to exist and be not invariant, fig. 2(d).

III(b) THEOREM. *In the pocket subset of $(l_1 f_1)$ (or $(l_n f_n)$) the arcs $M_1, M_2, \dots$, (or $M_n, M_{n-1}, \dots$) may be diverted in such a way that $M_1 \times (l_1 f_1) = 0$ (or $M_n \times (l_n f_n) = 0$) and yet the set remains independent.*

Proof: The theorem is obvious in case (1) above as there M_1 may merely follow the minor arc of Δ , $M_2, \dots, M_n$ being undisturbed. The other cases will be handled by I(d). Let $Q = (l_1 f_1)$, and construct the arcs $N_2, N_3, N_4, \dots$. Suppose $l_2, l_3, \dots$, are the last points common to successive links of $\gamma_1, \dots, \gamma_{n-1}$ the chain described in connection with the pocket above, and $f_2, f_3, \dots$, the first points common to the successive links of $\delta_1, \dots, \delta_{n-1}$ the chain of first links also used. As seen in the discussion of pocket subsets above one or other of these chains may fail to exist so that the corresponding set of points just mentioned reduces either to a or to b ; but as this circumstance has the effect of simplifying matters special treatment of such a case is not necessary. Now if not vacuous $Q_2 = N_2 \times M_2 = M_2 \times \{\Delta_{12}\} \dagger$ and so, $l_2 f_2$ being an arc of M_2 , $Q_2 \subset l_2 f_2 - l_2 f_2 \times (\Gamma_2 + \Delta_2)$, as otherwise, referring to III(a) and the supposition that $(l_1 f_1)$ is free, a contradiction arises. Therefore evidently l_2 precedes Q_2 , and Q_2 precedes f_2 . Likewise if $l_3 f_3 \subset M_3$ and $Q_3 \neq 0$, then $Q_3 \subset l_3 f_3 - l_3 f_3 \times (\Gamma_3 + \Delta_3)$, and so on. But by supposition $(l_1 f_1)$ is free and so $Q_n = 0$, hence some last Q , say $Q_k (k < n)$, is non-vacuous. Here also if arc $l_k f_k \subset M_k$, then $Q_k \subset l_k f_k - l_k f_k \times (\Gamma_k + \Delta_k)$, evidently with l_k preceding Q_k and Q_k preceding f_k . As the boundary of the

* By minor arc of M_{1n} will here be meant only that which does not contain both a and b ; by minor arc of Δ is meant that arc of it which with the minor arc of M_{1n} forms a simple closed curve not containing δ . These definitions can cause no confusion in this connection as a, l_1, f_1, b are all on M_1 .

$\dagger$ The symbol $\{\Delta_{12}\}$ has a meaning similar to $\{\Gamma_{kk+1}\}$ in the proof of I(d).

pocket with respect to F_{1n} is a subset of $\sum_{i=1}^n l_i + \sum_{i=1}^n f_i + p$ and as by I(e) for $2 \leq j \leq k$ $l_{j-1} + f_{j-1} \subset N_j \times M_{j-1}$ it is now clear that between l_{j-1} and f_{j-1} N_j is interior to the pocket. Moreover, since $Q_{k+1} = 0$, N_{k+1} skirts all the c -domains of F_{k+1} by way of M_k , and thus clearly does not leave the pocket between l_k and f_k . Between l_i and f_i let M_i coincide with N_{i+1} and be unchanged in position elsewhere ($1 \leq i \leq k$). The arc M_1 thus avoids $(l_1 f_1)$; $M_2, \dots, M_k$ remain independent of each other, are unchanged in position outside of the pocket, and are independent of $M_{k+1}, \dots, M_n$ which have remained unaltered. It is to be noticed that M_k may possibly now pass through the perhaps not invariant point p , fig. 2(d).

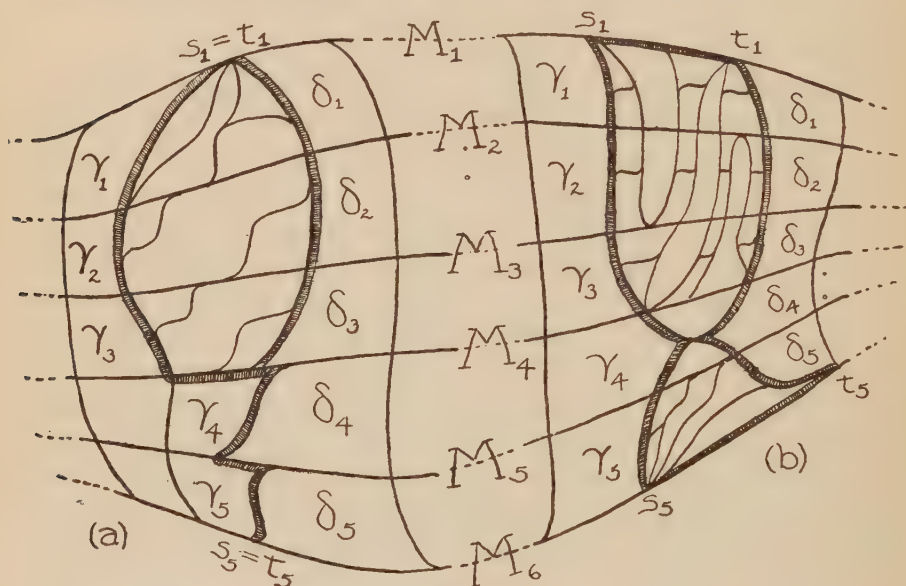


FIG. 3.

Switches $/s_n t_1/$ in heavy outline.

Cases 8 and 9 when they occur imply a condition of different significance. This will now be discussed.

Let C and D be a pair of corresponding intervals of type 8. There are on C end points of one or more chains of F_{1n} but all the opposite ends of these precede D on (M_n) , the last such point preceding being say s_n . On the other hand there is on (M_1) at least one end of a chain whose ends on (M_n) follow s_n for D is not free. Let t_1 be the first point of (M_1) which is an end of a chain among whose opposite ends are points following s_n . That is t_1 is

the first end of some chain $\delta_1, \dots, \delta_{n-1}$ with a -boundary arc $t_1 t_2 \dots t_{n-1} t_n$, t_n not preceding s_n but perhaps coinciding with it, whose links are the first ones possible satisfying the other requirements.

Two possibilities arise with regard to t_1 . (1) Perhaps t_1 is a last but not only end of a chain of last domains $\gamma_1, \dots, \gamma_{n-1}$ ($\gamma_1 \neq \delta_1$ evidently) with b -boundary arc $s_1 s_2 \dots s_{n-1} s_n$ ($t_1 = s_1$).* Here s_1 is not on C but other ends of this chain may be. This chain may have ends on (M_n) preceding s_n but none following. (See fig. 3(a)). (2) Otherwise t_1 is an only end or is not an end of chains whose first links precede δ_1 and whose last opposite end is s_n . Here as t_1 is not on C it must be preceded by some last end s_1 of a chain of last links $\gamma_1, \dots, \gamma_{n-1}$ and b -boundary arc $s_1 s_2, \dots, s_{n-1} s_n$ ($s_1 \neq t_1$), fig. 3(b). The segment $(s_1 t_1)$ is free. The supposition that t_1 is a limit of preceding end points of chains reduces easily to (1).

With regard to s_n there are also two possibilities as follows:

(1) $\Delta_{n-1} \times s_n b \supset \Delta_{n-1} \times M_n \supset s_n$ and $\neq s_n$ where $s_n b$ is an arc of M_n ; and (2) s_n precedes $\Delta_{n-1} \times M_n$ hence $(s_n t_n)$ is free, t_n being first point of $\Delta_{n-1} \times M_n$. The chains γ and δ can neither cross nor have a common link. The subset of F composed of $s_1 s_n$ (b -boundary arc of γ), $s_n t_n$ on M_n , $t_n t_1$ (a -boundary arc of δ), $t_1 s_1$ on M_1 , and the subset of F_{1n} , if any, which is included between them will be called a *switch subset* or *switch* of F_{1n} . The notation for it will be $/s_n t_1/$, s_1 and t_n being omitted altogether as of less importance even when distinct from s_n and t_1 . The bar $/$ indicates that C is a -joined, D b -joined, and makes the property of III(c) below easier to remember.

The discussion above was referred not only to the corresponding pair CD but also to M_1 and point a . A similar discussion for CD referring to point b and M_n determines a similar set but not necessarily the same one. If, however, the sets are not identical a particular pair of intervals (no longer corresponding) may be chosen instead of CD which will when discussed from M_n and b give the identical set $/s_n t_1/$. Similarly 9 on CD either from a and M_n or b and M_1 may be used to determine switches, the same or not, for which the symbol will be $\setminus s_1 t_n \setminus$, as above s_i preceding t_i ($i = 1, n$).

The most important property of the switch is the following:

* Although s_n itself is a point of the b -boundary arc of a chain with ends on C , $s_1 s_2 \dots s_{n-1} s_n$ is not necessarily such an arc; the chain determining it may have all its ends on (M_1) following C . It is a convenience however to introduce the symbol s_n at once, before indicating which of the b -boundary arcs of which it may be an end is the most important.

III(c) THEOREM. In the Switch $/s_nt_1/$ the arcs $M_2, M_3, \dots, M_n$ can be shifted from the points $s_2, s_3, \dots, s_n$ to the points $t_1, t_2, \dots, t_{n-1}$ without losing their independence, and in such a way that M_2 reaches M_1 first at t_1 and M_n leaves its original position for good at s_n . In the switch $\setminus s_1 t_n \setminus$ the arcs $M_1, M_2, \dots, M_{n-1}$ can be shifted from $s_1, s_2, \dots, s_{n-1}$ to $t_2, t_3, \dots, t_n$ without losing their independence, and in such a way that M_1 leaves its original position for good at s_1 and M_{n-1} reaches M_n at t_n and not before.

Only one of these statements will be proved as the procedure for proving the other is similar.

Proof: Refer to I(d) and I(e), and to prove the first let Q be the arc (at_1) of M_1 . Construct the arcs $N_2, \dots, N_n$. For either supposition concerning t_1 clearly the γ chain is one having an end in Q whereas no domain of the δ chain can be a member of a chain having an end in Q . Thus $N_2, \dots, N_n$ coincide with $M_2, \dots, M_n$ at the points $s_2, \dots, s_n$, but with $M_1, \dots, M_{n-1}$ at $t_1, \dots, t_{n-1}$. Let $M_2, \dots, M_n$ follow $N_2, \dots, N_n$ from $s_2, \dots, s_n$ to $t_1, \dots, t_{n-1}$. M_2 hence meets M_1 first at t_1 as t_1 is the first point of M_1 on the boundary of a c -domain of F_{12} which has no point of Q on its boundary. As s_n is the last end of any chain having an opposite end in Q , N_n and with it M_n as diverted leaves the original position of M_n at s_n . $N_2, \dots, N_n$ having been independent so are $M_2, \dots, M_n$ as now diverted.

In the next section of this proof it will be necessary to make various, even an infinite number of, switch and pocket constructions simultaneously. In what ways these constructions may affect each other is accordingly an essential problem. Various observations follow.

On Switches. (1) Any switch and its associated chains occupy on both (M_1) and (M_n) more than one point, for in $/s_nt_1/$ if $s_1 = t_1$ and $(as_1) \subset M_1$ then $\Gamma_1 \times (as_1) \neq 0$, and if $s_n = t_n$ and $(t_nb) \subset M_n$ then $\Delta_{n-1} \times (t_nb) \neq 0$.*

(2) Given $/s_nt_1/$, no point of (at_1) arc of M_1 is joined by any chain of F_{1n} to a point of (s_nb) arc of M_n . Also given $\setminus s_1 t_n \setminus$ no point of (at_n) is so joined to one of $(s_1 b)$. These follow from the nature of the sets.

(3) If among the point pairs $s_1, s'_1, t_1, t'_1, s_n, s'_n, t_n, t'_n$ one pair is non-coincident then $/s_nt_1/$ and $/s'_n t'_1/$ have at most n points and these all invariant in common. This is the argument for one case. Suppose t'_1 follows t_1 on (M_1) ; then s'_1 does not precede t_1 as $(s'_1 t'_1)$ is free. If s'_n were to precede t_n , then a point, t_1 , preceding t'_1 would be joined by a chain to a point,

* Notation here is as in the discussion immediately preceding III(c).

t_n , following s_n' which contradicts (2). But then the links of the chain γ' joining s_1' to s_n' follow or at most coincide with those of δ joining t_1 to t_n . Hence the switches have in common at most the points $s_i' = t_i$ ($i=1, 2, \dots, n$) which are invariant.

The same thing is true of $/s_n t_1/$ and $\setminus s_1' t_n \setminus$ if t_n' follows t_n , or s_n' follows s_n , or if s_1' precedes s_1 , or t_1' precedes t_1 .* When t_n' follows t_n the proof is this. As $(s_n' t_n')$ is free s_n' does not precede t_n , and if $s_n' = t_n$ then $t_n \neq s_n$ by observation (1). But then if s_1' precedes t_1 a point, namely s_1' preceding t_1 , is joined to a point, namely s_n' following s_n . This contradicts (2) and the statement follows as above.

(4) From (3) it appears that in any set of switches of the same type the switch construction may be made simultaneously as any two have in common only invariant points, the two sets of diverted arcs having in no wise interfered while changing position. Also from (3) it follows that two switches of different type when determined by pairs of corresponding intervals may have the switch construction made independently in both, as if $C D$ determines $/s_n t_1/$ and $C' D'$ following $C D$ determines $\setminus s_1' t_n \setminus$ then some chain from M_n must have ends on M_1 following C and preceding C' so $C \times C' = 0$, and as t_n does not follow D while t_n' follows D' t_n' follows t_n , even though D and D' reduce to coincident points.

(5) Only a or b can be the limit of an infinite set of switches, for among any infinite set must be an infinite subset of the same type and between any two of the same type is a chain; then apply I(c).

On Pockets. (1) In $/s_n t_1/$ if $(s_1 t_1)$ is free the switch contains a pocket, otherwise a given switch and pocket have only invariant common points as between the two lies a chain.

(2) Pockets of the first of the four sorts enumerated are simple and cause no difficulties. Given then a set of pockets made up of the other sorts. Such a set if infinite can have only a or b as set limit. If between any two of the set there is a chain then any two have at most invariant common points and in all of the sets simultaneously and without interference the pocket construction can be made, as it does not affect invariant points. What then is the relation between two pockets not having a chain between them? Evidently one must depend on a maximum free segment $(l_1 f_1)$ of (M_1) , the other on

* Other relations between switches of these types are possible and for these the statement is not necessarily true as may be seen by drawing a figure, but for using switches of these sorts fortunately no need arises.

one $(l_n'f_n')$ of (M_n) . But the γ chain of the first pocket precedes the second pocket and the γ' chain of the second precedes the first, so these two chains must coincide. Similarly the δ and δ' chains must coincide. Hence $l_j = l_j'$ ($j = 1, 2, \dots, n$) and $f_j = f_j'$. The common points of the b -boundary are $l_1l_2, \dots, l_{n-1}l_n$ of the γ chain and the a -boundary are $f_1f_2, \dots, f_{n-1}f_n$ of the δ chain are not invariant, as the two pockets have no chain between. If there is more than one such point the two pockets have no common point, if there is one such point they have it in common, if there is no such point actually there is but one pocket of sort (4) (see pocket classification), but it will be convenient still to look upon this set as two pockets. Now if $(l_1l_n) \times (f_1f_n)$, these symbols representing the chain boundary arcs in question, is vacuous or but a single point, evidently pocket constructions in each of the pockets may interfere, fig. 2(d). Such a pair of pockets as well as the free segments determining them will be said to be *opposite*. Evidently their sum is a switch of either type, $/l_nf_1/$ or $\setminus l_1f_n \setminus$.

(3) In section 4 pocket constructions will be used when available to divert M_1 or M_n from one or other but not both of the intervals in each pair of a given set of corresponding intervals. Among these intervals certain ones by being contained in maximum free segments, for example (l_1f_1) , will identify an associated set of such segments and pockets. Suppose in fact that $(l_1f_1) \supset C_{i_s} + \dots + C_u + \dots + C_{i_t}$ ($i_s \leq u \leq i_t$) but no others of the set, while (l_nf_n) the opposite of (l_1f_1) , if it exists, contains $D_{j_s} + \dots + D_v + \dots + D_{j_t}$ ($j_s \leq v \leq j_t$). The circumstances under which pocket constructions may and will be subsequently used to divert M_1 or M_n and the other arcs affected from one of each pair of corresponding intervals in the set is here summarised as condition (P).*

(P): Among the given set of corresponding intervals at least one of each pair is free, and one at least of the free ones of each pair is in a maximum free segment, as (l_1f_1) , which either (1) has no opposite among the pockets determined by the intervals of the set (that is (l_nf_n) fails to exist or fails to contain intervals of the set), or (2) has an opposite in which a pocket construction can be made independent of that in itself, or (3) has an opposite containing intervals corresponding only to those in itself or an opposite containing every interval to which an interval in itself corresponds, (that is $i_s \leq j_s$ and $j_t \leq i_t$, or $j_s \leq i_s$ and $i_t \leq j_t$).

* This condition can be made some what less restrictive but not without difficulty out of proportion to the benefit accruing.

These conditions not being satisfied, if $j_s < i_s$ and therefore $j_t < i_t$, the sum of the pockets will be taken as the switch $/l_n f_1/$ or $/s_n t_1/$; whereas if $i_s < j_s$ and hence $i_t < j_t$ the sum will be taken as $\setminus l_1 f_n \setminus$ or $\setminus s_1 t_n \setminus$.

The next section will be devoted to the proof of the theorem mentioned at the end of section 2.

4. THEOREM. *If none of the pairs of corresponding intervals of M_{1n} is of type 17 then F contains $n + 1$ independent arcs from a to b .*

Assuming that none of the pairs is of this type construction of the $n + 1$ arcs, to be designated as $L_1, L_2, \dots, L_n, L_{n+1}$, will be attempted. Convenience divides the argument into the following three cases; in case 1 it will be assumed that the arc of II(c) does not occur upon C_0 or D_0 , and that the entire set of corresponding intervals satisfies (P); in case 2 that the same arc is not on C_0 or D_0 but that (P) is not satisfied; in case 3 it will be assumed that the arc of II(c) is on C_0 .

Proof: Case 1. Let $G = I + K + M_{1n}$. The topological properties of G are precisely those of any set F_{jj+1} , in fact by a transformation of the type described in II(e) G is converted into such a set. Let $\sum_{i=-\infty}^{+\infty} C_i$ be the set of intervals of M_1 , and $\sum_{i=-\infty}^{+\infty} D_i$ those of M_n . Subsets $[C]$ and $[D]$ will now be chosen as follows from these. For each j let $[C]$ contain C_j if (1) C_j is free but D_j is not, or if (2) both C_j and D_j are free and the pocket determined by C_j either has no opposite in the set of pockets determined by $\sum_{i=-\infty}^{+\infty} D_i$ or has one in which the pocket construction may be made independently of one in itself or has one containing only intervals which correspond to intervals contained in itself; and let $[D]$ contain D_j if D_j is free and C_j is not or if both C_j and D_j are free and either the pocket determined by D_j has no opposite among those determined by $\sum_{i=-\infty}^{+\infty} C_i$ or has an opposite in which a pocket construction can be made independently of one in itself or an opposite containing only intervals corresponding to those in itself but not all of them. As the entire set of intervals satisfies (P) this analysis for each pair of intervals puts C_j into $[C]$, or D_j into $[D]$, or both. Moreover, pocket constructions may be made simultaneously in all of the pockets determined by all the intervals of $[C]$ and $[D]$, and independently.

Let the cross domains of G be $\dots \phi_{-p}, \dots, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2, \dots, \phi_q, \dots, (\phi_{-1} = \phi_0 \text{ if } K_0 = 0)$. From the definition of K and from II(e)

$\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1) = \sum_{i=-\infty}^{+\infty} C_i$ and $\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_n) = \sum_{i=-\infty}^{+\infty} D_i$. Let $[\Phi]$ be the subset of $\sum_{i=-\infty}^{+\infty} \Phi_i$ for which $\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1) = [C]$, then by the classification determining $[C]$ and $[D]$ $(\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_n) \subset [D]$. Following III(b) divert the arcs $M_1, \dots, M_n$ from the original positions of M_1 and M_n through the pockets associated with the intervals of $[C]$ and $[D]$. The resulting point sets are still independent by reason of an assumption made throughout this case, and are arcs owing to that property for pockets corresponding to (5) for switches. Call them $L_1, \dots, L_n$; evidently $F_{j-1, j+1} \supset L_j$ ($j = 2, 3, \dots, n-1$), $F_{12} \supset L_1$, $F_{n-1, n} \supset L_n$; and $L_1 \times [C] = 0$, $L_n \times [D] = 0$, $L_i \times L_j = a + b$ ($i \neq j$).

In G L_{n+1} will now be constructed. To do this use I(d) in G , $j = 1$ $i = 1$, and $Q = \sum_{i=-\infty}^{+\infty} C_i - [C]$. The set Q_2 , which is clearly on (M_n) not on (M_2) , will then, since $\sum_{i=-\infty}^{+\infty} C_i - [C] = (\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_1)$, be equal to the set $(\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_n)$, that is $Q_2 \subset [D]$. As N_2 constructed by I(d) is for any Q such that $N_2 \times (M_1) \subset \sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1)$ and $N_2 \times (M_n) \subset \sum_{i=-\infty}^{+\infty} \Phi_i \times (M_n)$; for this particular Q , $N_2 \times (M_1) = [C]$ and so $N_2 \times (L_1) = 0$; also $N_2 \times (M_n) \subset Q_2$, hence $N_2 \times (M_n) \subset [D]$ and so $N_2 \times (L_n) = 0$. Likewise $N_2 \times (\sum_{j=2}^{n-1} (L_j)) = 0$ as $N_2 \times I_{1n} = 0$ whereas $I_{1n} \supset \sum_{j=2}^{n-1} (L_j)$. The set N_2 is clearly an arc; let it be L_{n+1} . But then L_j ($j = 1, 2, \dots, n+1$) is a set of arcs in F from a to b contradicting the theorem in this case since independent and $n+1$ in number.

Case 2: The suppositions on this case imply at least one pair of corresponding intervals belonging to type 8, or to type 9, or to types 1 to 7 with one or both free and subsets of maximal free segments determining pockets having opposites among the set of pockets determined by $\sum_{i=-\infty}^{+\infty} C_i$ and $\sum_{i=-\infty}^{+\infty} D_i$, the pocket constructions in an opposite pair not being independent and the pair itself being such that, (a) $j_s < i_s$ and $j_t < i_t$, or (b) $i_s < j_s$ and $i_t < j_t$. Both 8 and the free case with (a) imply a switch of the sort $/s_n t_1/$ with s_n preceding D of the determining pair and t_1 following C ; the construction for these two cases is identical and hereafter the two will not be kept separate. Similarly 9 and the free case with (b) yield to the same treatment; in fact these two cases need not be considered at all since upon reflecting

F with respect to any straight line parallel to the straight line ab and then renumbering the arcs M_j in the reverse order, these two are the same as the first two. If a contradiction is obtained when a start is made with the first two one may similarly be derived for the other two. The procedure being the same whatever the corresponding interval pair upon which 8 or (a) occurs, suppose it to be $C_0 D_0$.

Let $/s_n t_1/$ be any switch determined by 8 or the free case with (a) in connection with $C_0 D_0$. As t_1 follows C_0 it follows some last interval C_{i-1} of M_1 , $i-1 \geq 0$; and as s_n precedes D_0 it precedes some first interval D_m of M_n , $m \leq 0$. Deflect L_1 from M_1 into K_{i_1} through a_{1i_1} ; in the switch $L_2, \dots, L_n$, whose construction is thus begun, move from $M_2, \dots, M_n$ to $M_1, \dots, M_{n-1}$, L_2 first reaching M_1 at t_1 while L_n leaves M_n at s_n . Finally from any convenient point of K_m introduce L_{n+1} upon M_n at b_{nm} . The construction of the $n+1$ independent arcs is now begun. Their continuation depends entirely upon the nature of the remaining pairs of intervals and for convenience various sub-cases will now be distinguished. As the situations met while constructing the $n+1$ arcs to a are no more general than those met toward b , only the construction to b will be completed as that to a follows the same lines.

The following possibilities call for separate treatment: (1) there is no pair of corresponding intervals subsequent to $C_{i-1} D_{i-1}$; (2) the next or some subsequent pair implies a switch $\setminus s_1' t_n \setminus$ with s_1' preceding C_{i_1} , t_1' following D_{i_1} ; (3) the next or some subsequent pair implies a switch $/s_n' t_1'/$ with s_n' preceding D_{i_1} and t_1' following C_{i_1} ; (4) neither the next nor any subsequent pair implies a switch; (5) there exist subsequent pairs which imply switches but none of these has the property explicitly added in (2) or (3).

Case 2 (1). Observe in this connection that t_1 is different from b ; this is obvious if the switch is determined by 8 while if determined by (a) $t_1 = b$ would mean $(s_1 b)$ on M_1 and thus $(s_n b)$ on M_n were both free and so $i_t = j_t$, a contradiction. But if $t_1 \neq b$, since no pair of corresponding intervals follows $C_{i-1} D_{i-1}$, $K_{i_1} = B \neq 0$. But L_1 being thus in B may be drawn through it to b , $L_2, \dots, L_{n+1}$ meanwhile following $M_1, \dots, M_n$ independently of each other and of L_1 to b , thus in this case completing the $n+1$ arcs to b .

Case 2 (2). Special circumstances attend the occurrence of this case also. Owing to note (2) on switches s_1' must not precede t_1 and so, since C_{i-1} precedes t_1 and C_{i_1} follows s_1' , $a_{1i_1} \neq b_{1i_1}$. This being the case proceed as follows. Let $\setminus s_1' t_n \setminus$ be the last such switch with s_1' preceding C_{i_1} and

let the last interval of a corresponding pair preceding t_n' be D_{l_2-1} , $l_2 > l_1$. In the switch $\backslash s_1' t_n' \wedge$ let the arcs $L_2, \dots, L_n$ drop back from $M_1, \dots, M_{n-1}$ to $M_2, \dots, M_n$; let L_1 return upon M_1 from K_{l_1} through b_{1l_1} the first point of C_{l_1} and thus independent of L_2 which leaves M_1 at a preceding point namely s_1' ; finally let L_{n+1} follow M_n to a_{nl_2} and there pass into K_{l_2} quite independent of L_n which first reaches M_n at t_n' subsequent to a_{nl_2} which is the last point

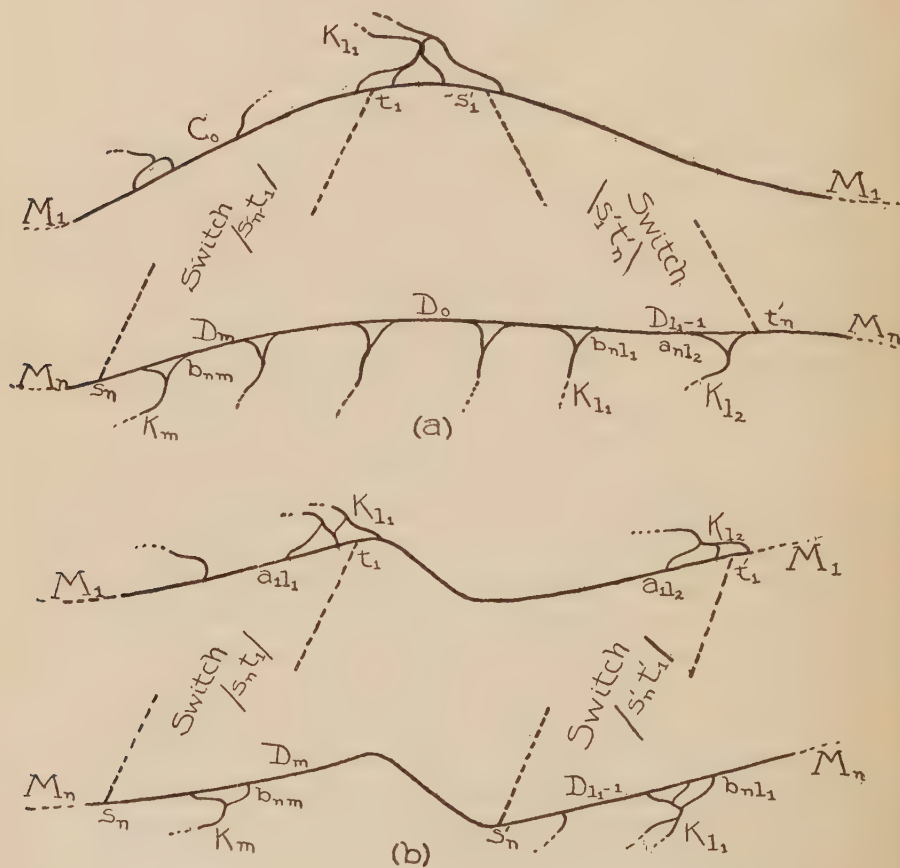


FIG. 4.

of D_{l_2-1} , fig. 4(a). The $n+1$ independent arcs have now been carried on into the next pair of corresponding intervals, or a pair subsequent to these.

Case 2 (3). This case is slightly more general than the two preceding as it imposes no conditions upon I or K . Two suppositions requiring somewhat different handling will be treated separately. As $/s_n' t_1' \wedge$ will at the outset be selected the last switch of this sort having s_n' before D_{l_1} . Suppose

s_n' does not precede D_m . Let C_{l_2-1} be the last interval of M_1 preceding t_1' , $l_2 > l_1$. The switch construction can be employed immediately. Let $L_3, \dots, L_{n+1}$ follow $M_2, \dots, M_n$ from their last points in the switch $/s_n t_1/$ to their first in $/s_n' t_1'/$, in so doing clearly they remain stationary or approach b owing to note (3) on switches, in particular are these statements true of L_{n+1} as s_n' does not precede D_m . Let them in $/s_n' t_1'/$ then be raised from $M_2, \dots, M_n$ to $M_1, \dots, M_{n-1}$, L_{n+1} leaving M_n at s_n' previous to point $b_{n l_1}$, L_3 reaching M_1 first at t_1' after $a_{1 l_2}$; take L_2 out into K_{l_2} through $a_{1 l_2}$; and finally bring L_1 in from K_{l_1} upon M_n through $b_{n l_1}$. $L_1, \dots, L_{n+1}$ have meanwhile remained independent and have been introduced upon the next or a subsequent pair of intervals, fig. 4 (b).

Before undertaking the next case the previous one must be completed for the assumption s_n' precedes D_m ; this may certainly occur as $/s_n t_1/$ was chosen as any switch having s_n before D_0 , t_1 after C_0 ; it can not occur if $/s_n' t_1'/$ is the last switch having s_n before D_0 , and consequently can not occur for $/s_n'' t_1''/$ after $/s_n' t_1'/$ nor $\setminus s_1'' t_n'' \setminus$ after $\setminus s_1' t_n' \setminus$ in constructing after cases 2 (2) and the first part of 2 (3) as $/s_n' t_1'/$ and $\setminus s_1' t_n' \setminus$ were selected as the last switches satisfying other requirements as imposed there. It is undesirable to pick $/s_n t_1/$ with the thought of avoiding this case as doing that makes the construction toward b less general than that toward a . However, the situation s_n' precedes D_m affects the fate of only one of the arcs involved and that arc is L_{n+1} . When the switch construction is made in $/s_n' t_1'/$ $L_1, \dots, L_n$ behave as above, but L_{n+1} instead of remaining where it is or approaching b must turn from b_{nm} toward a along M_n until it reaches s_n' and there be acted upon by the switch. Two difficulties seem to arise. It looks as if L_{n+1} might have to go as far as s_n before encountering s_n' for possibly s_n coincides with t_n ; the independence of the arcs would thus be lost as L_n already occupies s_n . This difficulty vanishes upon referring to note (1) on switches where it is seen that if $s_n = t_n$ then Δ_{n-1} has a point on M_n following t_n which s_n' can not precede. The other difficulty is this. Perhaps in the next step a switch $/s_n'' t_1''/$ will need to be used which has s_n'' also preceding D_m so that L_1 to reach it will have to approach a from $b_{1 l_1}$ thus encountering L_{n+1} at b_{nm} ; this is impossible because $/s_n' t_1'/$ was the last switch for which s_n' preceded D_{l_1} so $/s_n'' t_1''/$ must have s_n'' on or following D_{l_1} . Or perhaps a pocket required on D_{l_2} depends on a maximum free segment having an end not following b_{nm} so that L_1 might have to come to this point to enter the pocket construction thus colliding with L_{n+1} ; this is impossible owing to the situation on $C_{l_1} D_{l_1}$ for D_{l_1} either has a chain actually ending upon it (e. g. when C_{l_1}

D_{l_1} are of type 8) or is contained in a pocket which with its opposite forms a switch $/s_n't_1'/$ which is followed by a chain from t_n to t_1 , both which circumstances prevent any pockets on D_{l_2} having an end preceding D_{l_1} much less one not following b_{nm} . The $n+1$ arcs are thus successfully extended in such a way that subsequent construction along similar lines will not be interfered with.

Case 2 (4). The statement of this case indicates that among the pairs of corresponding intervals $C_{l_1}D_{l_1}$, $C_{l_1+1}D_{l_1+1}$, $\dots$ no pair is of types 8 or 9 and no pair contains only free intervals exhibiting (a) or (b) as described at the beginning of this section. Thus this set of intervals satisfies condition (P). Represent those of the set on M_1 by $\sum_{i=l_1}^{+\infty} C_i$ and those on M_n by $\sum_{i=l_1}^{+\infty} D_i$ and using the same criterion as in case 1 select from these two summations the subsets $[C]$ and $[D]$. Now in G with Q on M_1 equal to $\sum_{i=l_1}^{+\infty} C_i - [C]$ construct N_2 by I(d). Since for any Q on M_1 and N_2 in G , $N_2 \times (M_1) = \sum_{i=-\infty}^{+\infty} C_i$ and $N_2 \times (M_n) \subset \sum_{i=-\infty}^{+\infty} D_i$ and since $b_{l_1}b \times \sum_{i=-\infty}^{+\infty} C_i = \sum_{i=l_1}^{+\infty} C_i$ and $b_{n_{l_1}}b \times \sum_{i=-\infty}^{+\infty} D_i = \sum_{i=l_1}^{+\infty} D_i$ for this particular Q $N_2 \times (b_{l_1}b) = [C]$ and $N_2 \times (b_{n_{l_1}}b) = [D]$, $b_{l_1}b$ and $b_{n_{l_1}}b$ representing arcs of M_1 and M_n respectively which lack the end point b . But the intervals of $[C]$ and $[D]$ are subsets of maximal free segments of M_{1n} all following t_1 and t_n in the pockets determined by which the pocket construction can be made simultaneously and independently. Once the arcs $L_2, \dots, L_{n+1}$ can be brought to position at $t_1, t_2, \dots, t_{n-1}, t_n$ without points upon $[C]$ and $[D]$ they may be subjected to this construction and carried on independently to b in such a way that $L_2 \times [C] = 0$ and $L_{n+1} \times [D] = 0$. But the switch construction has already left $L_2, \dots, L_n$ at $t_1, \dots, t_{n-1}$ respectively, so the only question is as to the whereabouts of L_{n+1} . The arc L_{n+1} had been brought to M_n at b_{nm} , the first point of D_m and clearly preceding any interval of $[D]$, from K_m . If the first pocket required by $[D]$ is determined by the maximal free segment (pq) and $b_{nm} = p$ then construction may proceed without any trouble; if p follows b_{nm} then draw L_{n+1} along M_n until p is reached; if p precedes b_{nm} * draw L_{n+1} along M_n toward a from b_{nm} until it meets p . As q follows $b_{n_{l_1}}$, it follows b_{nm} , so no difficulties arise. The pocket construction can then be used and $L_2, \dots, L_{n+1}$ taken on independently to b , with $L_2 \times [C] = 0$ and

* Of course p does not precede t_n and if $s_n = t_n$ it can not coincide with t_n . Compare the situation here with that in the last paragraph of case 2 (3).

$L_{n+1} \times [D] = 0$. But N_2 in passing from the boundary of ϕ_{l_1} , whose limits on M_1 and M_n are C_{l_1} and D_{l_1} respectively, to the boundary of $\phi_{l_{n+1}}$ must pass through a point or limit point of K_{l_1} which is clearly not on L_2 or L_{n+1} . Now L_1 when last heard from was in K_{l_1} ; join it then in K_{l_1} to any convenient point of N_2 and let it follow N_2 to b . The construction of $n+1$ independent arcs has thus been in this case as in case 1 completed to b .

Case 2 (5). Let $C_r D_r$ be the first pair of corresponding intervals ($r > l_1$) which belong to types 8 or 9 or form with the pairs $C_{l_1} D_{l_1}$, $C_{l_{n+1}} D_{l_{n+1}}$, $\dots$, $C_{r-1} D_{r-1}$ a set not satisfying condition (P). The pair implies a switch of type $/s_n' t_1' /$ or $\setminus s_1' t_n' \setminus$, suppose in order to make the discussion slightly less abstract, and for this reason only as the argument for either case follows similar lines, that the switch is $/s_n' t_i' /$. Let the symbol represent the last switch for which s_n' precedes D_r , and suppose D_{l_2} is the first that s_n' precedes $l_1 < l_2 \leq r$. The pairs of corresponding intervals $C_{l_1} D_{l_1}$, $\dots$, $C_{l_{n+1}} D_{l_{n+1}}$ satisfy the condition (P). Form the subset $[C]$ of $\sum_{i=l_1}^{l_2-1} C_i$ and the subset $[D]$ of $\sum_{i=l_1}^{l_2-1} D_i$. Let $Q = \sum_{i=l_1}^{l_2-1} C_i - [C]$ and proceed precisely as in (4) to construct N_2 . When N_2 is constructed join L_1 to it from K_{l_1} and let L_1 follow N_2 toward b until it reaches the first limit point of K_{l_2} , that is necessarily a_{1l_2} or a_{nl_2} , and then carry L_1 into K_{l_2} , out of the way. Meanwhile as in 2 (4) bring $L_2, \dots, L_{n+1}$, with precisely the same circumspection as regards L_{n+1} , on until they arrive at the switch $/s_n' t_1' /$ having on the way avoided all the intervals of $[C]$ and $[D]$ and thus independent of L_1 . In the switch then execute exactly the same maneuver as in case 2 (3). The $n+1$ independent arcs are thus introduced upon an interval pair subsequent to the pair $C_{l_{n+1}} D_{l_{n+1}}$.

In summary upon case 2 notice that in subcases (1) and (4) the problem is complete, while in the other cases the work paused at the point where the next remark would be, "there are now five possibilities," seen at a glance to be identical with the five just handled for the interval pair $C_{l_1} D_{l_1}$ and disposable in essentially the same way for the pair, say, $C_{l_2} D_{l_2}$, even although here they may follow a switch construction $\setminus s_1' t_n' \setminus$ rather than merely $/s_n t_1 /$. So continue the process. Either eventually a pair of intervals is reached exhibiting case 2 (1) or (4) at which the construction ends, or it runs on through an infinite number of stages bringing $n+1$ independent arcs indefinitely near to b . To see that in the limit these arrive as independent arcs at b consult II(d), note (5) on switches, and the first few sentences of (2) on pockets.

Accordingly if the limit points of J are not on C_0 (or D_0) one pair of corresponding intervals is of type 17, as otherwise from a to b in F there are $n + 1$ independent arcs.

Case 3. Suppose then K_0 is vacuous* and the limits of J occur on C_0 . Omit J from F . Then $F - J$ contains on M_{1n} a pair of intervals of type 17. Restore J to the set. It will now be shown that if in $F - J$ the intervals $C_0 D_0$ are the only corresponding ones of type 17 and that all the limits of the joining chains on C_0 occur between the extreme limits of J so that D_0 is no longer joined to either of the intervals into which as mentioned in II(e) J divides C_0 , then a contradiction arises—there are $n + 1$ arcs from a to b in F . For assume these conditions. For J the notation attached to K_0 will be used as far as it applies. There is at least one chain from D_0 to $(a_{10} b_{10})$ and from any convenient point of that constituent of J which connects a_{10} and b_{10} and any convenient points invariant and n in number one from each of the arcs $M_1, \dots, M_n$ and common to the successive links of such a chain $n + 1$ arcs will now be constructed. When the extra arc arrives in $K_m, m > 0$, troubles cease, as no subsequent corresponding intervals are of type 17 and so previous methods apply.

Now D_0 is not free. Suppose C_0 is free. If the pocket containing it has no opposite among the pockets determined by the intervals $\sum_{i=0}^{+\infty} D_i$ or has an opposite for which $i_t > j_t$, pocket construction will be used depending upon the maximum free segment (pq) p preceding b_{10} , q following C_0 , in fact following some last interval C_{m-1} . From the points indicated in the last paragraph extend $L_1, \dots, L_n$ and let them be diverted from $M_1, \dots, M_n$ only when in the pocket (pq) in which case let them follow its pocket construction; they are thus independent and $L_1 \times (pq) = 0$. From the chosen point of J bring L_{n+1} in through b_{10} upon M_1 which let it follow to a_{1m} last point of C_{m-1} from which it passes into $K_m (m > 0)$. Farther construction now follows previous lines.

If C_0 is not in a pocket, or if when in one is in one which has an opposite with $i_t < j_t$ then a switch $\setminus s_1 t_n \setminus$ is determined with s_1 on $(a_{10} b_{10})$ and t_n following $D_{m-1} (m > 0)$. This switch drops the arcs $L_1, \dots, L_{n-1}$ from the points at which they are begun to $M_2, \dots, M_n$ L_1 leaving M_1 at s_1 , L_{n-1} reaching M_n at t_n ; let meanwhile L_n leave M_n at a_{nm} for $K_m (m > 0)$ and let

* If K_0 is not vacuous the limits of J must occupy a subinterval of $a_{10} b_{10}$ or of $a_{n0} b_{n0}$ owing to the nature of the set K_0 , see section 2.

L_{n+1} come in from J upon M_1 through b_{10} . As the $n + 1$ st arc is now in K_m instead of J nothing farther need be said.

The procedure toward a is similar, it being noted that although J redivides C_0 into C_{-1} and C_0 it does not reseparate D_{-1} and D_0 which still coincide. Thus there are $n + 1$ arcs in F from a to b in this case also unless a pair of corresponding intervals is joined.

5. THEOREM I. F possesses n complementary domains, of which one may be its unbounded complementary domain, $\lambda_{12}, \lambda_{23}, \dots, \lambda_{n-1n}, \lambda_{n1}$, where λ_{jj+1} is a subset of I_{jj+1} and λ_{n1} is exterior to M_{1n} , such that:

(1) Δ_{jj+1} has at least one point in common with Δ_{j-1j} and at least one in common with Δ_{j+1j+2} but none whatever in common with any of the others unless possibly a or b (subscripts reduced mod n), ($j = 1, 2 \dots n - 1$).

(2) all the points common to Δ_{j-1j} and Δ_{jj+1} are on the arc M_j ,

(3) at least one of the common points of Δ_{j-1j} and Δ_{jj+1} is on (M_j) , that is, is different from a and from b ; this point will be designated as p_j .

Proof: Any chain joining two corresponding intervals plus the domain of whose boundary these intervals is a subset form a set of domains satisfying these conditions.

Let any one of the points of $\Delta_{j-1j} \times \Delta_{jj+1} \times (M_j)$ be called p_j .

THEOREM II. $F - \sum_{i=1}^n p_i$ consists of two mutually separated sets F_a and F_b where $F_a \supset a$ and $F_b \supset b$.

Proof: Let c_{jj+1} be a point of λ_{jj+1} . As any boundary point of a complementary domain of a continuous curve is accessible from any point of the domain there is an arc $(p_j c_{jj+1} p_{j+1})$ in λ_{jj+1} . Draw all n of these arcs. As they are finite in number, join only at their ends, and join in an order determined by part (1) of theorem I above, the sum $\sum_{j=1}^n p_j c_{jj+1} p_{j+1}$ is a simple closed curve C , having only the points $\sum_{i=1}^n p_i$ in common with F . Moreover a and b are separated by this curve for suppose both are interior. Then b must be interior to one of the following simple closed curves with mutually

exclusive interiors $a m_1 p_1 c_{12} p_2 m_2 a$, $a m_2 p_2 c_{23} p_3 m_3 a$, $\dots$, $a m_{n-1} p_{n-1} c_{n-1n} p_n m_n a$, $a m_n p_n c_{n1} p_1 m_1 a$ where m_j is some point of M_j between a and p_j . Suppose for instance that it lies in $a m_1 p_1 c_{12} p_2 m_2 a$. Then M_j ($j=3, 4, \dots, n$) must to reach b from a cross $a m_1 p_1$, or $p_1 c_{12} p_2$, or $p_2 m_2 a$, none of which it may do; or else lie already within M_{12} entirely, which the preliminary ordering forbids. Similarly it is proved that not both a and b lie outside the simple closed curve. Thus F consists of two parts F_a and F_b , $F_a \supset a$ and $F_b \supset b$, one exterior to C and one interior, and of course the points $\sum_{i=1}^n p_i$. The theorem is thus seen to be proved.

THEOREM III. $E - \sum_{i=1}^n p_i$ is the sum of two mutually separated sets E_a and E_b where $E_a \supset a$ and $E_b \supset b$.

Proof: Suppose not. Then a and b are points of some maximal connected subset of $E - \sum_{i=1}^n p_i$ in which there can be drawn an arc W from one to the other. This arc cuts C in some point p , as a is separated from b by C . But p is not a point of F as it is not a point of $\sum_{i=1}^n p_i$. Proceed along W from p toward a and toward b until reaching the first points p_a and p_b in these directions which W has in common with F , these must be distinct as W is an arc, and must exist as a and b are in F and F is closed. But then $(p_a p p_b)$ is a part of some maximal connected subset of $E - F$ which has at least two limit points, for example p_a and p_b , in F . This is a contradiction.*

COROLLARY I. Under the hypotheses of the general theorem E possesses a set of complementary domains having the property of Theorem I above.†

THEOREM IV. If among the subsets of the Continuous Curve E whose omission from the continuous curve separates it into two mutually separated sets E_a and E_b , $E_a \supset a$ and $E_b \supset b$ where a and b are given points, is one consisting of n distinct points but none consisting of fewer than n , then there is a set of n independent arcs in E from a to b and no set of independent arcs of E from a to b consists of more than n .

* W. L. Ayres, "Arc Curves and Basic Sets of a Continuous Curve."

† For F has such a set and then see W. L. Ayres, "Concerning Arc-curves and Basic sets of a Continuous Curve."

Proof: Obviously if the set of independent arcs from a to b in E were $n + 1$ in number, $F - \sum_{i=1}^n p_i$ would not be separated between a and b , as at least one of the $n + 1$ arcs would fail to contain any point of $\sum_{i=1}^n p_i$, so in this case $\sum_{i=1}^n p_i$ could not separate E in this way either. On the other hand, if the arc curve were to contain at most $n - j$ ($0 < j \leq n$) independent arcs from a to b , then by the general theorem there would be $(n - j) < n$ points whose omission would separate E between a and b into two mutually separated sets one containing a and the other b . As this contradicts hypothesis, the arc curve of $a + b$ with respect to E contains for some construction n independent arcs from a to b but for no construction more than n .

THEOREM V. *Under the hypotheses of theorem IV there is a set of n complementary domains of E related to any chosen set of n separating points in the manner of the domains of theorem I and the points $\sum_{i=1}^n p_i$.*

Proof: For by theorem IV the arc curve of $a + b$ in E contains a set of n independent arcs but no set of more than n . Let a set of n be $M_1, \dots, M_n$ ordered as heretofore and for each i let p_i be on (M_i) , $\sum_{i=1}^n p_i$ being the chosen separating set; evidently one point of this set must be on each of the arcs M_i and no more than one can be. Now the set F_{jj+1} as previously described must be cut between a and b by the omission from it of $\sum_{i=1}^n p_i$ as it is a subset of F containing a and b and hence must also be cut by the omission of the points p_j and p_{j+1} which are the only ones of $\sum_{i=1}^n p_i$ which it contains. But then as before p_j and p_{j+1} are on the boundary of some bounded complementary domain of F_{jj+1} , hence also one of F like λ_{jj+1} and finally one of E . The set of domains thus determined is easily seen to possess the properties of those in theorem I.

THEOREM VI. *Either under the hypotheses of the general theorem or under the hypotheses of theorem IV, there is a simple closed curve C whose interior contains either a or b and such that $C \times E = \sum_{i=1}^n p_i$ where $\sum_{i=1}^n p_i$ is any set of n points whose omission divides E into two mutually separated sets one containing a and the other b .*

Proof: As the sets of n points dividing the curve E are identical with those dividing the curve F , the theorem follows upon the application of theorem V and the process of theorem II. It is to be observed that C can not enclose a or b chosen arbitrarily as the set of domains of theorem V may possibly not include the unbounded complementary domain of E .

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

